Before starting this activity, you should already have calculated the APR in the simplest case: where a sum of money is borrowed at a particular time and paid back, with interest, in a single repayment at a later date using the following formula:

$$C = \frac{A}{(1+i)^n}$$

which gives the APR, i, as a decimal when a loan of £C is paid back by a single repayment £A after n years.

This activity deals with calculating APR in more difficult cases, when the repayment will be made in more than one instalment. You will be able to check the value of the APR and also calculate it.

A Checking the given APR for a loan paid back in several annual instalments

Information sheet

In cases when the repayment will be made in more than one instalment, the single repayment formula you have already met is adapted to include all the instalments.

When there are four annual instalments, A_1, A_2, A_3 and A_4, the formula becomes

$$C = \frac{A_1}{1+i} + \frac{A_2}{(1+i)^2} + \frac{A_3}{(1+i)^3} + \frac{A_4}{(1+i)^4}$$

You may need to calculate the APR for different numbers of instalments.

The general formula for any number m instalments is given below.

General APR formula for cases involving more than one instalment

$$C = \sum_{k=1}^{m} \left(\frac{A_k}{(1+i)^{\left(t_k\right)}} \right)$$

where i is the APR expressed as a decimal

k is the number identifying a particular instalment

A_k is the amount of instalment k

t_k is the interval in years between the payment of the instalment and the start of the loan.

Think about

- What is the formula for 3 annual instalments?
- What is the formula for 5 annual instalments?

When you are asked to check a given APR value, just substitute this into the formula to check whether it gives the correct value of the loan, C.

Example

A loan of £5000 is repaid in 3 equal annual instalments of £2000.
The APR is quoted as 9.7%. Is this correct?

For 3 annual instalments, the formula is

$$C = \frac{A_1}{1+i} + \frac{A_2}{(1+i)^2} + \frac{A_3}{(1+i)^3}$$

Substituting $i = 0.097$ and $A_1 = A_2 = A_3 = 2000$ gives:

$$C = \frac{2000}{1.097} + \frac{2000}{1.097^2} + \frac{2000}{1.097^3} = £1823.15 + £1661.95 + £1514.99$$

$$= £5000.09$$

As this is very near £5000, the statement that the APR is 9.7% is likely to be correct.

Think about…

- What does the symbol $\sum$ mean?
- What do the values £1823.15, £1661.95 and £1514.99 represent?
- How could you be *certain* that 9.7% is correct to 1 decimal place?
  (Hint. What are the smallest and largest values that 9.7, correct to 1 dp, can be? Try substituting these in the formula and see how the values of C compare with £5000)

Try these

A Checking the given APR for a loan paid back in several annual instalments

1 Check the value of the APR given in each of the following advertisements:

Borrow £10 000. Repay in 2 annual instalments of £6000 and £7200 **APR 20%**	Borrow £3600. Repay in 2 annual instalments of £2250 and £2025 **APR 12.5%**

2 A borrower repays a debt of £4000 in 3 equal annual instalments of £1500. Show that the APR is 6.1% correct to 1 decimal place.

3 A loan of £12 000 is repaid in 4 equal annual instalments of £4000. Show that the APR is 12.6% correct to 1 decimal place.

4 A debt of £11 000 is repaid in 4 annual instalments of £2000, £3000, £4000 and £5000. Show that the APR is 9.0% correct to 1 decimal place.

B Finding a value for the APR

Information sheet

You may be asked to find a value for the APR (i) to a given degree of accuracy.

When there is only one repayment, the formula for C can be rearranged to find a value for i relatively easily – you should already have used this idea.

However, when there is more than one instalment, it is not possible to do this. Finding a value for the APR becomes much more difficult.

One method that can be used in such cases is called the **interval bisection method**. This involves starting with a range of possible values for i, then repeatedly halving this range until the range becomes so small that it is possible to give an accurate value for i and hence the APR.

If you require a very accurate value for i, the method is time-consuming and tedious to carry out by hand. The more instalments there are, the worse it gets. The use of a computer makes this much less onerous.

The next example explains how the method works, using the relatively simple case where a loan is repaid by 4 instalments. (In real life things can become far more complex than this.)

Finding the APR using the interval bisection method

This 'trial and improvement' method is summarised below.

Choose an interval of values within which you believe the APR lies.

Substitute the i value at each end of your interval into $\boldsymbol{C} = \sum_{k=1}^{m}\left(\frac{A_k}{(1+i)^{(t_k)}}\right)$

to check that the range you have chosen does include the value of i.

Next use the mid-point of your range as i. Find C using $\boldsymbol{C} = \sum_{k=1}^{m}\left(\frac{A_k}{(1+i)^{(t_k)}}\right)$

If the calculated value of $\boldsymbol{C}$ is **too low**, this implies that the value used for $\boldsymbol{i}$ was **too high**. You now know that the correct value of $\boldsymbol{i}$ **lies in the lower half of the interval you used.**

If the calculated value of $\boldsymbol{C}$ is **too high**, this implies that the value used for $\boldsymbol{i}$ was **too low**. You now know that the correct value of $\boldsymbol{i}$ **lies in the upper half of the interval you used**.

Repeat the last two steps, using the new interval within which you know the correct value of i lies. The fact that this is half of the previous interval gives this method its name of '**interval bisection**'.

Repeat the process again and again, until the interval is narrow enough to give you an accurate value of i. The number of steps this takes will depend on how accurate you want your value for the APR to be.

Example

A loan of £24 000 is repaid in 4 annual instalments of £7500, £7500, £8000 and £8000. Use interval bisection to find the APR correct to 1 decimal place.

Solution

Suppose we start by trying the interval 6% to 12%.

When $i = 0.06$, $$C = \frac{7500}{1.06}+\frac{7500}{1.06^2}+\frac{8000}{1.06^3}+\frac{8000}{1.06^4} = 26\,804$$

When $i = 0.12$, $$C = \frac{7500}{1.12}+\frac{7500}{1.12^2}+\frac{8000}{1.12^3}+\frac{8000}{1.12^4} = 23\,454$$

As the true value of C, £24 000, lies between these values, then the true value of i must lie between 0.06 and 0.12.

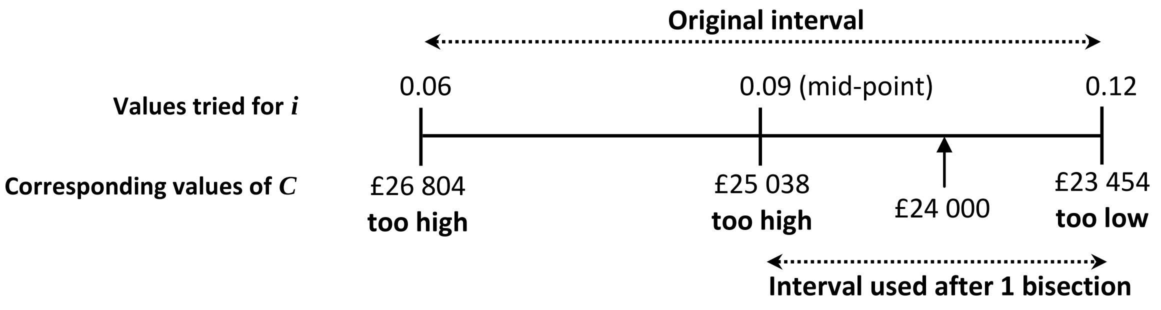
1st bisection

The mid-point of the interval $0.06 < i < 0.12$ is 0.09

When i = 0.09, $C = \dfrac{7500}{1.09} + \dfrac{7500}{1.09^2} + \dfrac{8000}{1.09^3} + \dfrac{8000}{1.09^4}$ = 25 038

This is too high, indicating that 0.09 is too low, so i must lie between 0.09 and 0.12.

This sketch illustrates the results so far:



2nd bisection

The mid-point of $0.09 < i < 0.12$ is 0.105

When i = 0.105, $C = \dfrac{7500}{1.105} + \dfrac{7500}{1.105^2} + \dfrac{8000}{1.105^3} + \dfrac{8000}{1.105^4}$ = 24 225

This is still too high, so 0.105 is too low and i must lie between 0.105 and 0.12.

3rd bisection

The mid-point of $0.105 < i < 0.12$ is 0.1125

When i = 0.1125, $C = \dfrac{7500}{1.1125} + \dfrac{7500}{1.1125^2} + \dfrac{8000}{1.1125^3} + \dfrac{8000}{1.1125^4}$ = 23 834

This is too low, so 0.1125 is too high and i must lie between 0.105 and 0.1125.

This means that the APR must lie between 10.5% and 11.25%.

Hence the **APR = 11% to the nearest %.**

For a more accurate value of the APR, the process can be continued:

4th bisection

The mid-point of $0.105 < i < 0.1125$ is 0.10875

When i = 0.10875, $C = \frac{7500}{1.10875} + \frac{7500}{1.10875^2} + \frac{8000}{1.10875^3} + \frac{8000}{1.10875^4}$ = 24 028

This is too high, so 0.10875 is too low, so i must lie between 0.10875 and 0.1125.

5th bisection

The mid-point of $0.10875 < i < 0.1125$ is 0.110625

When i = 0.110625,

$C = \frac{7500}{1.110625} + \frac{7500}{1.110625^2} + \frac{8000}{1.110625^3} + \frac{8000}{1.110625^4}$ = 23 931

This is too low, so 0.110625 is too high and i must lie between 0.10875 and 0.110625.

6th bisection

The mid-point of $0.10875 < i < 0.110625$ is 0.1096875

When i = 0.1096875,

$C = \frac{7500}{1.1096875} + \frac{7500}{1.1096875^2} + \frac{8000}{1.1096875^3} + \frac{8000}{1.1096875^4}$ = 23 980

This is too low, so 0.1096875 is too high and i must lie between 0.10875 and 0.1096875.

7th bisection

The mid-point of $0.10875 < i < 0.1096875$ is 0.10921875

When i = 0.10921875,

$C = \frac{7500}{1.10921875} + \frac{7500}{1.10921875^2} + \frac{8000}{1.10921875^3} + \frac{8000}{1.10921875^4}$ = 24 004

This is slightly too high, so 0.10921875 is too low and i must lie between 0.10921875 and 0.1096875.

Think about
Why is it still not possible to give the APR correct to 1 decimal place?

8th bisection

The mid-point of the range $0.10921875 < i < 0.1096875$ is 0.109453125

When i = 0.109453125,

$$C = \frac{7500}{1.109453125} + \frac{7500}{1.109453125^2} + \frac{8000}{1.109453125^3} + \frac{8000}{1.109453125^4} = 23992$$

This is slightly too low, so 0.109453125 is too high.
Therefore i must lie between 0.10921875 and 0.109453125.

This means that the APR must lie between 10.92% and 10.945%.

Hence the **APR = 10.9% to 1 decimal place.**

Think about

- Why is it now possible to give the APR correct to 1 decimal place?
- In the questions below, how will you choose the initial values of i for the first interval bisection?
- How will you know when you have carried out enough interval bisections?

Try these

B Finding a value for the APR

1 A borrower repays a debt of £4000 in 2 annual instalments of £2300.

a Show that the APR lies between 8% and 12%.

b Use the interval bisection method to find the APR to the nearest %.

2 A loan of £8000 is repaid in 3 equal annual instalments of £3000.

a Show that the APR lies between 8% and 10%.

b Use the interval bisection method to find the APR correct to 1 decimal place.

3 A debt of £12 500 will be repaid in 4 annual instalments of £4000, £4250, £4500 and £4750.

a Show that the APR lies between 10% and 20%.

b Use the interval bisection method to find the APR correct to 1 decimal place.

4 A loan of £6000 is repaid in 3 annual instalments of £2000, £2500 and £3000.

a Use the interval bisection method to find the APR to the nearest %.

b Continue to use the interval bisection method until you find the value of the APR correct to 1 decimal place.

5 A debt of £3000 is repaid in 4 equal annual instalments of £1000.

a Use the interval bisection method to find the APR to the nearest %.

b Continue to use the interval bisection method until you find the value of the APR correct to 1 decimal place.

6 A borrower repays a loan of £7500 by paying £5000 after one year, and £4000 after another year. Find the APR correct to 1 decimal place.

7 To repay a loan of £5500, a borrower pays annual instalments of £1000, £1500, £2000 and £2500. Find the APR correct to 1 decimal place.

8 A lender offers a choice of two ways of repaying a loan of £10 000.

Repayment method A: Repay in 3 annual instalments of £4000, £4500 and £5000.

Repayment method B: Repay in 4 equal annual instalments of £3500.

a Find the total amounts repaid by each method.

b Calculate the APR in each case, correct to 1 decimal place.

c i Give a reason why a borrower may prefer to use Method A to repay the loan of £10 000.
ii Give a reason why a borrower may prefer to use Method B to repay the loan of £10 000.

Reflect on your work

- Why is it important to know the APR when a loan is repaid in several instalments?
- What methods do you know for calculating the APR?
  Write down the formulae that are used when there is 1 repayment, 2 repayments, and 3 repayments.
- How would you set up a spreadsheet to carry out the calculations required by the interval bisection method?